An Introduction to Granular Flow

K. Kesava Rao
Indian Institute of Science

Prabhu R. Nott
Indian Institute of Science
## Contents

2.7 The Mohr's Circle for the Two-Dimensional Stress Tensor 65
2.8 The Relation Between the Coulomb and Mohr-Coulomb Yield Conditions 68
2.9 Active and Passive States of Stress and the Value of the Janssen K-Factor 69
2.10 Shear Tests 70
  2.10.1 The Critical State 73
  2.10.2 The Hvorslev Surface 75
  2.10.3 The Roscoe Surface 77
  2.10.4 The Yield Surface 77
2.11 Yield Surfaces in \( \sigma_1-\sigma_2-\upsilon \) Space 79
2.12 Yield Loci in the \( \sigma_1-\sigma_2 \) and \( \sigma-\tau \) Planes 81
2.13 Flow Rules 82
  2.13.1 The Lévy–Mises and the Prandtl–Reuss Equations 84
  2.13.2 The Coaxiality Condition 87
  2.13.3 The Plastic Potential 91
  2.13.4 Positive Dissipation 91
  2.13.5 Associated and Nonassociated Flow Rules 92
2.14 Equations for Plane Flow 93
  2.14.1 The Mohr’s Circle for the Rate of Deformation Tensor 93
  2.14.2 The Coaxiality Condition 94
  2.14.3 The Flow Rule 95
  2.14.4 Implications of the Associated Flow Rule 96
  2.14.5 Rowe's Stress–Dilatancy Relation 96
  2.14.6 Summary of the Governing Equations for Plane Flow 100
2.15 The Relation Between Yield Loci in the \( N-T \) and \( \sigma-\tau \) Planes 100
2.16 The Double-Shearing Model 101
2.17 Summary 105

3 Flow through Hoppers 116
  3.1 Experimental Observations 116
    3.1.1 Flow Rate 116
    3.1.2 Kinematics 121
    3.1.3 Solids Fraction Profiles 122
  3.2 Theory for Steady, Plane Flow 123
    3.2.1 The Critical State Approximation 123
  3.3 The Smooth Wall, Radial Gravity (SWRG) Problem 125
  3.4 The Effect of Wall Roughness 131
  3.5 Solutions with Allowance for Rough Walls and Vertical Gravity 134
    3.5.1 The Brennen–Pearce Solution 134
    3.5.2 The Radial Stress and Velocity Fields 138
    3.5.3 Linearized Stability Analysis 141
    3.5.4 Downward Integration from the Radial Fields 143
    3.5.5 The Successive Approximation Procedure 145
  3.6 A Re-Examination of the Exit Condition 149
  3.7 An Alternative Exit Condition 152
  3.8 The Smooth Wall, Radial Gravity Problem for Compressible Flow 154
  3.9 Summary 160

4 Flow through Wedge-Shaped Bunkers 166
  4.1 Experimental Observations 166
### Contents

4.1.1 Flow Regimes 166  
4.1.2 Kinematics 166  
4.1.3 Wall Stresses 175  
4.1.4 Bins 178  
4.2 Models for Bunker Flow 182  
4.2.1 The Bin Section 183  
4.2.2 The Transition Region 190  
4.2.3 The Hopper Section 203  
4.3 Summary 208

**5 Theory for Slow Three-Dimensional Flow** 213  
5.1 Constitutive Equations Involving a Yield Condition 213  
5.1.1 The Yield Condition 213  
5.1.2 Symmetry Considerations 220  
5.1.3 Conventional Triaxial Tests 220  
5.1.4 Isotropic Compression Tests 222  
5.1.5 Compression Tests 223  
5.1.6 Cubical Triaxial Tests 225  
5.1.7 Comparison of Yield Conditions with Data 226  
5.1.8 Flow Rules 226  
5.1.9 Data Related to Flow Rules 228  
5.1.10 Steady, Fully Developed Flow of a Rigid-Plastic Material 229  
5.1.11 One-Dimensional Deformation of a Rigid-Plastic Material 230  
5.2 Constitutive Equations That Do Not Involve a Yield Condition 233  
5.2.1 Hypoelastic and Hypoplastic Models 233  
5.2.2 Some Features of (5.66) 235  
5.2.3 Steady, Fully Developed Flow of a Hypoelastic Material 236  
5.2.4 One-Dimensional Deformation of a Hypoelastic Material 236  
5.3 Summary 239

**6 Flow through Axisymmetric Hoppers and Bunkers** 249  
6.1 Experimental Observations 249  
6.1.1 Flow Rate 249  
6.1.2 Velocity Profiles 249  
6.1.3 Density Profiles 251  
6.1.4 Flow Patterns 253  
6.1.5 Stress Profiles 256  
6.2 Theory for Steady, Axisymmetric Flow Through a Hopper 259  
6.2.1 The Haar-von Karman Hypothesis 261  
6.2.2 The Radial Stress and Velocity Fields for the Mohr-Coulomb Yield Condition and the Haar-von Karman Hypothesis 264  
6.2.3 The Drucker-Prager Yield Condition and Levy's Flow Rule 266  
6.2.4 Comparison of Predicted and Measured Velocity Profiles 269  
6.2.5 Criteria for Mass Flow 269  
6.3 A Hybrid Hypoplastic-Viscous Model 273  
6.4 The Kinematic Model for Batch Discharge from a Bin 276  
6.5 Summary 280

**7 Theory for Rapid Flow of Smooth, Inelastic Particles** 285  
7.1 Preliminaries and Scaling 285
7.1.1 Model for Inelastic Collisions 288
7.1.2 Hydrodynamic Description of Rapid Granular Flows 289
7.2 Heuristic Hydrodynamic Theory for High-Density Flows 290
7.2.1 Application to Uniform Plane Shear 292
7.3 Kinetic Theory for a Granular Gas of Smooth Inelastic Particles 295
7.3.1 Statistical Preliminaries 295
7.3.2 The Evolution of \( f^{(1)} \) 297
7.3.3 The Equilibrium Distribution Function 300
7.3.4 The Departure from Equilibrium 301
7.3.5 Maxwell Transport Equation 302
7.3.6 The Equations of Motion 305
7.3.7 The Chapman-Enskog Expansion 307
7.3.8 Constitutive Relations at Leading Order 310
7.3.9 Distribution Function at \( O(K) \) 311
7.3.10 Solution for \( \Phi_K \) 316
7.3.11 Constitutive Relations at \( O(K) \) 319
7.3.12 Distribution Function and Constitutive Relations at \( O(\epsilon) \) 321
7.3.13 Constitutive Relations to First Order in \( K \) and \( \epsilon \) 323
7.4 Anisotropy of the Microstructure 325
7.5 Extension to Granular Mixtures 326
7.6 Summary and Discussion 328

8 Analysis of Rapid Flow in Simple Geometries 331
8.1 Boundary Conditions at Solid Walls 331
8.1.1 Heuristic Theory 332
8.1.2 Kinetic Theory 334
8.2 Plane Couette Flow 339
8.2.1 Predictions of the High-Density Theory 340
8.2.2 Some Features of the High-Density Solutions 345
8.2.3 Predictions of the Kinetic Theory 346
8.3 Flow in Inclined Chutes 349
8.3.1 Some Experimental Observations of Chute Flow 351
8.3.2 Analysis of Steady, Fully Developed Flow 355
8.3.3 High-Density Theory 356
8.3.4 Some Features of the High-Density Solutions 358
8.3.5 Predictions of the Kinetic Theory 359
8.4 Stability of Rapid Shear Flows 366
8.4.1 Stability of Unbounded Plane Shear Flow 366
8.4.2 Stability of Plane Couette Flow 369
8.5 Summary 371

9 Theory for Rapid Flow of Rough, Inelastic Particles 374
9.1 Collision Models for Rough Particles 375
9.2 Equations of Motion for a Granular Gas of Rough, Inelastic Spheres 378
9.3 The Velocity Distribution Function 382
9.3.1 Nearly Elastic, Nearly Perfectly Rough Particles 382
9.3.2 Nearly Elastic, Nearly Smooth Particles 386
9.4 Constitutive Relations up to First Order in \( K, \epsilon, \) and \( \varepsilon \) 389
9.4.1 Nearly Elastic, Nearly Perfectly Rough Particles 389
# Contents

G.2 Frame Indifferent Scalars, Vectors, and Tensors

- G.2.1 Scalars 441
- G.2.2 Vectors 441
- G.2.3 Second-Order Tensors 442

G.3 The Principle of Material Frame Indifference 443

G.4 An Alternative Interpretation of a Change of Frame 443

Appendix H: The Evaluation of Some Integrals 449

- H.1 Integration Over k 449
- H.2 Integration Over k for Boundary Conditions 450
- H.3 Change of Variables 451
- H.4 Volume Integrals 452
- H.5 Gaussian Integrals 453

Appendix I: A Brief Introduction to Linear Stability Theory 454

Appendix J: Pseudo Scalars, Vectors, and Tensors 456

Appendix K: Answers to Selected Problems 459

References 463

Index 483